

# Types of Matrices, Determinants and Their Properties: A Comprehensive Study

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## Abstract

This paper presents a comprehensive exploration of matrix theory, focusing on the classification of matrices, determinant computation, and fundamental properties. Matrices serve as essential mathematical structures in linear algebra with widespread applications across physics, engineering, computer science, and economics. This study systematically examines various matrix types including square, rectangular, diagonal, symmetric, and special matrices, followed by an in-depth analysis of determinants and their algebraic properties. The paper establishes theoretical foundations while demonstrating practical computational methods, providing a rigorous treatment suitable for advanced undergraduate and graduate-level study in mathematics and related disciplines.

**Keywords:** Matrix theory, Linear algebra, Determinants, Matrix classification, Symmetric matrices, Matrix properties

## Introduction:

### Background and Motivation

Matrix theory forms the cornerstone of modern linear algebra and serves as an indispensable tool in numerous scientific and engineering disciplines. A matrix, defined as the systems of Linear equations can be easily organised in a rectangular arrangement or systematic matrix representation with respect to the number of Rows and Columns. Linear transformations, and multivariate data structures [1]. The systematic study of matrices dates back to ancient Chinese mathematics, with significant developments occurring in the 19th century through the work of Arthur Cayley and James Joseph Sylvester [2].

The importance of matrix theory extends far beyond pure mathematics. In physics, matrices represent quantum mechanical operators and describe rotations in three-dimensional space. Engineers utilize matrices for structural analysis, control systems, and signal processing. Computer scientists employ matrix operations in graphics rendering, machine learning algorithms, and network analysis. Economists apply matrix methods to input-output models and optimization problems [3].

### Scope and Objectives:

This paper aims to provide a comprehensive treatment of matrix classification and determinant theory. The primary objectives are:

- 1.To systematically classify matrices based on structural properties and special characteristics
- 2.To establish rigorous definitions and properties of determinants
- 3.To explore computational methods for determinant evaluation
- 4.To examine the relationship between matrix properties and determinant behavior
- 5.To demonstrate practical applications of these theoretical concepts

### Fundamental Definitions and Notation:

#### 2.1 Basic Matrix Concepts

**Definition 2.1** Matrix - a set of numbers distributed into m rows and n columns is referenced as a matrix A with an order  $m \times n$ . This is the way it is presented

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The term  $a_{ij}$  yields the location of an element in the i-th row and j-th column [4].

**Definition 2.2** (Square Matrix): A matrix is square if its number of rows equals its number of columns, or  $m = n$ . The order of a square matrix is  $n \times n$  or simply  $n$  [5].

**Definition 2.3** (Matrix Equality): Matrix Equality states that two matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$  of the same order are equal for all i and j if and only if  $a_{ij} = b_{ij}$ .

#### 2.2 Matrix Functions:

**Matrix Addition:** The addition of two matrices A and B of the same order  $m \times n$  is denoted by the matrix function  $(A + B)_{ij} = a_{ij} + b_{ij}$ .

**Multiplying scalars:** For a scalar k and matrix A:

$$([kA])_{ij} = k \cdot a_{ij}$$

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**Matrix Multiplication:** For  $A (m \times p)$  and  $B (p \times n)$ :

$$(AB)_{ij} = \sum_{k=1}^p a_{ik} b_{kj}$$

**Transpose:** As the Antithesis of  $A = [a_{ij}]$  of the dimension of the matrix is  $A^T = [a_{ji}]$ , the number of rows has been transposed into the Number of Columns  $[6]$ .

**3.1 The first classification of all Matrices are Classes based on 'Dimensions' :**

**3.1.1 Row Matrices (Row vectors)-** are Matrices with 1 Row (1xn).

An example would be  $A = [1 \ 2 \ 5 \ -6]$  (1x4 Row Matrix).

**3.1.2 Column matrices (Column vectors) :** A simply column vectors, are matrices defined as having only one column (i.e., a single column of m rows).

Therefore, for example,  $B = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$  is an example of a (2x1) column matrix .

**3.1.3 Rectangular Matrix:** Any matrix which has m not equal to n can be described as a rectangular matrix.

**Example:**  $C = \begin{bmatrix} 1 & 2 & 9 \\ 8 & 7 & 4 \end{bmatrix}$  ( $2 \times 3$  rectangular matrix)

**3.2 Algebraic Classification of Matrices with Definition and Examples:**

Sr.no.	Matrix Type	Definition	Example
1.	Matrix of Rows	A matrix with a single row	$A = [7 \ 4 \ 9]$
2.	Matrix of Columns	A matrix with just one column	$B = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$
3.	Rectangular Matrix	A matrix with rows $\neq$ columns	$C = \begin{bmatrix} 9 & 8 & 1 \\ 7 & 5 & 3 \end{bmatrix}$ ( $2 \times 3$ )
4.	Matrix Diagonal	A square matrix with zero values for all non-diagonal entries	$D = \begin{bmatrix} 9 & 2 \\ 6 & 1 \end{bmatrix}$
5.	Matrix of Identity	A matrix having 1s in the diagonal and 0s elsewhere	$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
6.	Symmetric Matrix	A Square matrix in which, $A = A^T$	$A = \begin{bmatrix} 6 & 2 \\ 2 & 8 \end{bmatrix}$
7.	Skew - Symmetric Matrix	A square matrix where the diagonal entries of the Matrix are 0 and $A^T = -A$	$A = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$
8.	Hermitian Matrix	A complex matrix satisfying $A = A$ (conjugate transpose)*	$A = \begin{bmatrix} 2 & 3-i \\ 3+i & 5 \end{bmatrix}$
9.	Matrix of Upper Triangular	Every element below the diagonal equals zero	$U = \begin{bmatrix} 2 & 7 \\ 0 & 5 \end{bmatrix}$
10.	Matrix of Lower Triangular	Every element above the diagonal equals zero	$L = \begin{bmatrix} 6 & 0 \\ 5 & 9 \end{bmatrix}$
11.	Orthogonal Matrix	A square matrix that is real and satisfies $A^T A = I$	$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

**Characteristics of matrices:**

**3.3.1 Diagonal Matrix :**

- Two diagonal matrices added together are diagonal
- When two diagonal matrices are multiplied , the result is diagonal
- When multiplied ,diagonal matrices commute

**3.3.2 Identity Matrix :**

- $AI = IA = A$  for any compatible matrix  $A$

- $I$  is unique for each dimension
- $\det(I) = 1$

### 3.3.3 Symmetric Matrix :

- A real symmetric matrix has real eigenvalues
- Different eigenvalues correspond to orthogonal eigenvector
- Every symmetric matrix is diagonalizable

### 3.3.4 Skew-Symmetric Matrix :

- Every diagonal component is zero
- $\det(A) = 0$  if  $n$  is odd
- Eigenvalues are either zero or entirely fictitious

### 3.3.5 Matrix of Upper and Lower Triangular :

- The product of the diagonal elements is the determinant
- When two upper (lower) triangular matrices are multiplied the result is upper (lower) triangular

## A triangular matrix's 4. Determinants: Definition and Computation

- inverse is likewise triangular

### 4.1 Historical Development

The concept of determinants emerged in the 17th century through the work of Leibniz, though systematic theory was developed later by Cauchy and Jacobi. Determinants provide a scalar value associated with square matrices that encodes important information about linear transformations [7].

### 4.2 Definition of Determinant

**Definition 4.2.1:** The determinant of a square matrix of order  $n$  is the scalar value  $\det(A)$  or  $|A|$ , that is recursively calculated :

For  $n = 1$ :  $\det([a_{11}]) = a_{11}$

For  $n = 2$ :  $\det \begin{pmatrix} A_{11} & A_{12} \\ B_{21} & B_{22} \end{pmatrix} = A_{11} B_{22} - A_{12} B_{21}$

For  $n \geq 3$ : Using cofactor expansion along row  $i$ :

$$\det(A) = \sum_{j=1}^n a_{ij} C_{ij}$$

where  $C_{ij} = (-1)^{i+j} \cdot M_{ij}$  is the cofactor, and  $M_{ij}$  is the minor (determinant of the  $(n-1) \times (n-1)$  submatrix obtained by deleting row  $i$  and column  $j$ ) [8].

### 4.3 Computation Methods

#### 4.3.1 Direct Computation (2x2 and 3x3 Matrices)

**2x2 Matrix :**

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

**3x3 Matrices :**

$$\det \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ B_{21} & B_{22} & B_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = A_{11} [(B_{22} C_{33}) - (B_{23} C_{32})] - A_{12} [(B_{21} C_{33}) - (C_{31} B_{23})] + A_{13} [(B_{21} C_{32}) - (C_{31} B_{22})]$$

## 5. Properties of Determinants

### 5.1 Fundamental Properties

**Property 5.1** (Determinant of Transpose):  $\det(A^T) = \det(A)$

This property implies that row and column operations have symmetric effects on determinants [9].

**Property 5.2** (Multiplicative Property): The Determinant of a Product is equal to the Product of the Determinants of the two matrices. This is the basic property that establishes a multiplicative homomorphism between the set of matrices and the set of scalars [10].

**Property 5.3** (Inverse Determinant): For a non-singular matrix  $A$ ,  $\det(A^{-1}) = 1/\det(A)$ .

**Property 5.4** (Scalar Multiplication): The Determinant of  $kA$  is equal to the Product of  $k$  and the Determinant of  $A$  for an  $n \times n$  matrix  $A$  and for any scalar  $k$  [11].

**Property 5.5** (Identity Determinant): The Determinant of the Identity Matrix,  $I$ , is equal to 1.

### 5.2 Properties Related to Row and Column Operations :

**Property 5.6** The Sign of the Determinant Changes when Two Rows (or Columns) are Interchanged:  $\det(A') = -\det(A)$ .

**Property 5.7** The Determinant of a Matrix  $A$  is 0 if either two Rows or Columns are (identically or proportionately)  $> 0$ .

**Property 5.8** If we add one row to another row using a factor 'k' we do not change the determinant of the matrix. The new determinant is equal to the old determinant. (i.e.,  $\det(A1) = \det(A)$ ) The above property can be derived by using the matrix procedure for  $\text{Row}(i)$  to  $\text{Row}(i) + k * \text{Row}(j)$  [11].

**Property 5.9** When we multiply determinant  $A$  by the number 'k' when we multiply determinant  $A1$  by the same scale factor of 'k'. So if we multiply a row by a constant factor, then the determinant of the row's matrix has also changed by that same constant.

### 5.3 Properties of Special Matrices

**Property 5.10** (Diagonal Matrix): For a diagonal matrix  $D = \text{diag}(d_1, d_2, \dots, d_n)$ :  $\det(D) = d_1 \cdot d_2 \cdot \dots \cdot d_n$

**Property 5.11** (Triangular Matrix): For upper or lower triangular matrix  $T$ :  $\det(T) = t_{11} \cdot t_{22} \cdot \dots \cdot t_{nn}$

**Property 5.12** (Orthogonal Matrix): For orthogonal matrix :  $\det(Q) = \pm 1$  for  $Q$

**Property 5.13** (Symmetric Matrix): Symmetric matrices' eigenvalue decomposition makes computation easier, yet they share the same determinant characteristics as general matrices.

**Property 5.14** (Skew-Symmetric Matrix): For skew-symmetric matrix  $A$  of odd order:  $\det(A) = 0$

For even order  $n = 2m$ :  $\det(A) = [\text{Pf}(A)]^2$

## 6. Determinants and Linear Systems

**Property 6.1** (Cramer's Rule) : To solve for the  $i$ th variable in the system  $Ax = b$ , where  $|A| \neq 0$ , we can replace the  $i$ th column of  $A$  with  $b$ . Thus,  $x_i = |A_i| / |A|$  [12].

**Property 6.2** ( System in Solvability):

A system's ability to find a unique solution exists only when  $\det(A) \neq 0$ , meaning that there exists no other solution that has a unique solution for this matrix. Thus, the only cases where a unique solution exists are when  $\det(A) \neq 0$  (i.e., either no solution or an infinite number of solutions exist).

### Applications/Importance

#### 7.1 Solve Linear Systems in Determinants

Linear Systems Determinants show theoretical characteristics (the existence and uniqueness of solutions through  $\det(A) \neq 0$ ) and computational techniques (Cramer's Rule) to solve linear systems, although very few modern numerical techniques for determining the solution of large systems use determinants as a better alternative [13].

#### 7.2 Matrix Invertibility

The condition  $\det(A) \neq 0$  is equivalent to  $A$  being invertible, making determinants essential for characterizing non-singular matrices. The inverse formula:  $A^{-1} = (1/\det(A)) \cdot \text{adj}(A)$

where  $\text{adj}(A)$  is the adjugate matrix, directly employs determinants [14].

### Conclusion

#### 8.1 Summary of Key Results

This paper has presented a comprehensive treatment of matrix classification and determinant theory. We systematically examined:

1. **Matrix Classification:** Matrices were categorized by dimension (row, column, square, rectangular), symmetry properties (symmetric, skew-symmetric, Hermitian), structural characteristics (diagonal, triangular, identity), and special properties (orthogonal).
2. **Determinant Theory:** We established rigorous definitions of determinants, explored multiple computation methods ranging from direct formulas to row reduction, and analyzed computational complexity.
3. **Determinant Properties:** Fundamental properties including multiplicativity, effects of row operations, behavior under transposition, and special cases for structured matrices were thoroughly investigated.
4. **Applications:** The practical significance of determinants in solving linear systems, testing invertibility, computing eigenvalues, and geometric transformations was demonstrated.

### Concluding Remarks

The study of matrices and determinants remains vibrant and essential across mathematics, science, and engineering. While classical theory is well-established, new applications continually emerge, particularly in data science, quantum information, and computational physics. The elegant mathematical structure of matrix theory continues to provide powerful tools for modeling and solving complex problems in the modern computational era.

The development of diverse types of matrices and the comprehensive theory of determinants creates the foundational knowledge required for the different fields of study for all types of researchers and practitioners. The expansion of available computational resources and the creation of new areas of study will lead to the further adaptation and implementation of these classical ideas as the relationship between linear algebra and contemporary fields of science and technology continues to evolve.

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### Conflicts of interest

The authors declare that there are no conflicts of interest regarding the of this paper

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